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# ELLIPTIC SINGULAR PERTURBATIONS OF FIRST ORDER DIFFERENTIAL OPERATORS VANISHING AT AN INTERIOR SURFACE

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# Elliptic singular perturbations of first order differential operators vanishing at an interior surface

by

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ABSTRACT

With the maximum principle asymptotic estimates are made for a class of linear elliptic singular perturbation problems with resonant turning-point behaviour in one or more independent variables.

KEY WORDS & PHRASES: *maximum principle, singular perturbation, asymptotic approximation, turning-point*

# 1. INTRODUCTION

We consider the Dirichlet problem for a function $\phi(x_1,\ldots,x_k,y_1\ldots y_m;\varepsilon)$ satisfying the linear elliptic differential equation

$$(1.1)\qquad L_\varepsilon\phi \equiv \varepsilon L_2\phi + L_1\phi = f(x,y;\varepsilon) \quad \text{in} \quad \Omega$$

with boundary conditions

$$(1.2)\qquad \phi = h(x,y;\varepsilon) \quad \text{on} \quad \partial\Omega\,.$$

The domain $\Omega$ is a bounded domain in $\mathbb{R}^n$ $(n=k+m)$ of such a form that

$$(1.3)\qquad (x,y) \in \Omega \quad \text{implies} \quad (x,0) \in \Omega.$$

The operator $L_2$ is a second order partial differential operator, uniformly elliptic in $\Omega$, $L_1$ is a first order differential operator; both operators have coefficients being Hölder continuous in $\Omega$,

$$(1.4)\qquad L_2 \equiv \sum_{i,j=1}^{k} \alpha_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^{k}\sum_{j=1}^{m} 2\beta_{ij}\frac{\partial^2}{\partial x_i\partial y_j} + \sum_{i,j=1}^{m} \gamma_{ij}\frac{\partial^2}{\partial y_i\partial y_j},$$

and

$$(1.5)\qquad L_1 \equiv \sum_{j=1}^{m} b_j\frac{\partial}{\partial y_j}.$$

The differential operator

$$(1.6)\qquad \sum_{i,j=1}^{k} \alpha_{ij}(x,0)\frac{\partial^2}{\partial x_i\partial x_j}$$

is supposed to be uniformly elliptic in $\Gamma_x = \{x \mid (x,0) \in \Omega\}$. Furthermore, it is assumed that

$$(1.7)\qquad b(x,y) = 0 \quad \text{iff} \quad |y| = 0,$$

$$(1.8)\qquad \nu(x,y)\cdot b(x,y) \le 0 \quad \text{on} \quad \partial\Omega,$$

$$(1.9)\qquad \sum_{j=1}^{m} b_j(x,y)y_j \le -L|y|^2 \quad \text{in} \quad \bar{\Omega},$$

where $\nu(x,y)$ is the outer normal to $\partial\Omega$, L a positive constant independent of $\varepsilon$ and $|y|$ the Euclidian length of y ($|y| = \sqrt{y_1^2+\ldots+y_m^2}$).

Condition (1.7) gives the problem its turning-point character; the coefficients of the first derivatives of (1.1) vanish on the set

$$(1.10)\qquad \Gamma = \{(x,y) \mid (x,y) \in \Omega, \quad |y| = 0\}$$

in the interior of the domain $\Omega$. Because of (1.9) we cannot apply the maximum principle in a way as carried out by DE JAGER [5], where k = m = 1 and $b_y(x,0) > 0$. For application of the maximum principle to elliptic singular perturbation problems without turning-points we refer to ECKHAUS and DE JAGER [2]. The class of elliptic problems with isolated turning-points was analyzed on its spectral properties by DE GROEN [4] and with formal methods by GRASMAN and MATKOWSKY [3].

It is our purpose to give asymptotic estimates of solutions of (1.1) - (1.9), which too are based on the maximum principle for elliptic differential equations. It turns out that the behaviour of the solution in $\bar{\Omega}$ is mainly determined by the elliptic problem in $\Gamma_x$ as described by (1.6).

## 2. MAXIMUM PRINCIPLE AND BARRIER FUNCTIONS

For the operator $L_\varepsilon$ given by (1.1), (1.4) and (1.5) we formulate the maximum principle as follows: a twice continuously differentiable function $\phi$ satisfying $L_\varepsilon\phi > 0$ in a domain $\Omega$ cannot have a maximum in $\Omega$, see PROTTER and WEINBERGER [8, p.61]. The following lemma is a direct consequence of the maximum principle.

<u>LEMMA 2.1</u>. *If the twice continuously differentiable functions* $\phi$ *and* $\Psi$ *satisfy*

(2.1) $$|L_\varepsilon\phi| < -L_\varepsilon\Psi \quad in \quad \Omega,$$

*and if* $|\phi| \le \Psi$ *on* $\partial\Omega$, *then* $|\phi| \le \Psi$ *in* $\bar{\Omega}$.

PROOF. From the maximum principle and (2.1) we deduce that $\phi - \Psi$ does not have a maximum in $\Omega$ and since $\phi - \Psi \le 0$ on $\partial\Omega$, we conclude that $\phi - \Psi \le 0$ in $\bar{\Omega}$. Similarly, $-\phi - \Psi$ does not have a maximum in $\Omega$, and $-\phi - \Psi \le 0$ at $\partial\Omega$, so that $-\phi - \Psi \le 0$ in $\bar{\Omega}$. Combining these results we obtain $|\phi| \le \Psi$ in $\bar{\Omega}$. □

The uniform boundedness with respect to $\varepsilon$ of solutions of (1.1) - (1.9) is established by the following theorem.

THEOREM 2.1. *Let the twice continuously differentiable function* $\phi$ *satisfy*

(2.2) $$|L_\varepsilon\phi| \le M|y|^2 \text{ in } \Omega$$

*and* $|\phi| \le N$ *on* $\partial\Omega$ *with* M *and* N *independent of* $\varepsilon$. *Then there exists a constant* K *independent of* $\varepsilon$ *such that*

(2.3) $$|\phi| \le K \text{ in } \bar{\Omega}.$$

PROOF. We introduce a so-called barrier function

(2.4) $$\Psi(x,y) = -U(x) + R|y|^2 + S,$$

in which we choose $R > M/L$ with L given by (1.9) and U(x) such that

(2.5) $$\sum_{i,j=1}^{k} \alpha_{ij}(x,0)\frac{\partial^2 U}{\partial x_i \partial x_j} = P + 2R \sum_{i,j=1}^{m} \gamma_{ij}(x,0) \text{ in } \Gamma_x$$

and $U = 0$ on $\partial\Gamma_x$, where P is some positive constant independent of $\varepsilon$. This boundary value problem for U has a unique twice continuously differentiable solution, see [1, p.336].

Since the coefficients $\alpha_{ij}$ and $\gamma_{ij}$ are Hölder continuous, there exists a positive constant F, such that

$$\sum_{i,j=1}^{k} \alpha_{ij}(x,y) \frac{\partial^2 U}{\partial x_i \partial x_j} - 2R \sum_{i,j=1}^{m} \gamma_{ij}(x,y) > F \text{ in } \Omega. \tag{2.6}$$

For $|y|^2 \geq \varepsilon F/M$ we have

$$-L_\varepsilon \Psi = \varepsilon\left\{ \sum_{i,j=1}^{k} \alpha_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} - 2R \sum_{i,j=1}^{m} \gamma_{ij} \right\} - 2R \sum_{j=1}^{m} b_j y_j > \tag{2.7}$$

$$-\varepsilon F + 2RL|y|^2 \geq M|y|^2.$$

Because of the Hölder continuity of $\alpha_{ij}$ and $\gamma_{ij}$ at $y = 0$, we may replace (2.6) by the following estimate for $|y|^2 < \varepsilon F/M$

$$\sum_{i,j=1}^{k} \alpha_{ij}(x,y) \frac{\partial^2 U}{\partial x_i \partial x_j} - 2R \sum_{i,j=1}^{m} \gamma_{ij}(x,y) = P + \delta(x,y) \tag{2.8}$$

with $\delta(x,y) \to 0$ as $\varepsilon \to 0$. Thus, for $|y|^2 < \varepsilon F/M$ we have

$$-L_\varepsilon \Psi = \varepsilon\{P+O(1)\} + 2RL|y|^2 > 2M|y|^2 > M|y|^2. \tag{2.9}$$

Finally, S of (2.4) is taken sufficiently large such that

$$\Psi \geq N \quad \text{on} \quad \partial\Omega. \tag{2.10}$$

From (2.7) and (2.9) we conclude that

$$|L_\varepsilon \phi| < -L_\varepsilon \phi \quad \text{in} \quad \Omega, \tag{2.11}$$

while from (2.10) it follows that $|\phi| \leq \Psi$ on $\partial\Omega$. Using lemma 2.1 we obtain $|\phi| \leq \Psi$ in $\bar{\Omega}$. Since the function $U(x)$ as well as the domain $\Omega$ is bounded, a positive constant K can be found such that $\Psi \leq K$ in $\bar{\Omega}$, which completes the proof of the theorem. □

The next theorems show that the barrier function only needs to satisfy the differential equation upto $O(\varepsilon)$ in a $O(\sqrt{\varepsilon})$ neighbourhood of $\Gamma$.

THEOREM 2.2. *Let the twice continuously differentiable function* $\phi$ *satisfy*

$$(2.12) \qquad |L_\varepsilon \phi| \le M\varepsilon \quad in \quad \Omega$$

*and* $|\phi| \le N$ *on* $\partial\Omega$ *with* M *and* N *positive constants independent of* $\varepsilon$. *Then there exists a constant* K *independent of* $\varepsilon$ *such that*

$$(2.13) \qquad |\phi| \le K \quad in \quad \bar{\Omega}.$$

PROOF. As a barrier function we introduce the function

$$(2.14) \qquad \Psi(x,y) = -U(x) + S,$$

with U satisfying

$$(2.15) \qquad \sum_{i,j=1}^{k} \alpha_{ij}(x,0) \frac{\delta^2 U}{\delta x_i \delta x_j} = M + 1$$

in $\Gamma_x$ and $U = 0$ on $\partial\Gamma_x$. Because of property (1.3) $\Psi$ is defined everywhere in $\Omega$. We see that

$$(2.16) \qquad |L_\varepsilon \phi| \le M\varepsilon < (M+1)\varepsilon = -L_\varepsilon \Psi.$$

Taking S sufficiently large we also have $|\phi| \le N \le \Psi$. Inequality (2.13) follows then from lemma 2.1. □

The proof of the following theorem as well as that of its corollary is the same as the proof of theorem 2.1 and will be omitted for that reason.

THEOREM 2.3. *Let the twice continuously differentiable function* $\phi$ *satify*

$$(2.17) \qquad |L_\varepsilon \phi| \le M(|y|^2 + \varepsilon) \quad in \quad \Omega$$

*and* $|\phi| \le N$ on $\partial\Omega$ *with* M *and* N *positive constants independent of* $\varepsilon$. *Then a constant* K *independent of* $\varepsilon$ *exists such that* $|\phi| \le K$ *in* $\bar{\Omega}$.

COROLLARY 2.1. *Let the twice continuously differentiable function* $\phi(x,y;\varepsilon)$ *satisfy*

$$(2.18) \qquad |L_\varepsilon\phi| \le M(|y|^2+\varepsilon)f(\varepsilon) \quad \textit{in} \quad \Omega$$

*and* $|\phi| \le Nh(\varepsilon)$ *on* $\partial\Omega$ *with* f *and* h *continuous, positive functions for* $0 < \varepsilon \le \varepsilon_0$ *(*$\varepsilon_0$ *sufficiently small) and with* M *and* N *independent of* $\varepsilon$*. Then a constant* K *independent of* $\varepsilon$ *exists such that in* $\bar{\Omega}$

$$(2.19a) \qquad |\phi| \le Kf(\varepsilon) \quad \textit{if} \quad h/f \textit{ is bounded for } \varepsilon \to 0,$$

*or*

$$(2.19b) \qquad |\phi| \le Kh(\varepsilon) \quad \textit{if} \quad f/h \textit{ is bounded for } \varepsilon \to 0.$$

## 3. ASYMPTOTIC APPROXIMATION

Let us assume that by some matched asymptotic approximation procedure we have found a formal uniformly valid asymptotic solution, say $\phi_{as}$, of (1.1), (1.2). Its validity can be proved as follows. Substitution of

$$(3.1) \qquad \phi(x,y;\varepsilon) = \phi_{as}(x,y;\varepsilon) + R(x,y;\varepsilon)$$

into (1.1) and (1.2) gives the following problem for R

$$(3.2) \qquad L_\varepsilon R = f - L_\varepsilon\phi_{as} \quad \text{in} \quad \Omega$$

$$(3.3) \qquad R = h - \phi_{as} \quad \text{on} \quad \partial\Omega.$$

If we are able to show that the right-hand sides of (3.2) and (3.3) have the desired behaviour to apply corollary 2.1, the smallness of the remainder term R is established. It is to expect that the problem (1.1) - (1.9) has a boundary layer structure. This complicates the construction of a suitable $\phi_{as}$, as the derivatives of $\phi_{as}$ may increase with some inverse power of $\varepsilon$ in the boundary layer. This difficulty can be surmounted by adding (small) boundary layer corrections to the asymptotic approximation.

As an example we deal with the following problem. Let the function $\phi(x,y;\varepsilon)$ satisfy

$$(3.4) \qquad L_\varepsilon \phi \equiv \varepsilon\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\} - y\frac{\partial\phi}{\partial y} = 0$$

in a bounded region $\Omega$ of $\mathbb{R}^2$ specified as

$$(3.5) \qquad \Omega = \{(x,y) \mid -1 < x < 1, -p_-(x) < y < p_+(x)\}$$

with $p_\pm(x) \in C^3[-1,1]$, $p_\pm(-1) = p_\pm(1) = 0$ and $p_\pm(x) > 0$ for $-1 < x < 1$, see figure 1. Moreover, $\phi$ satisfies the boundary conditions

$$(3.6) \qquad \phi(x,p_\pm(x)) = h_\pm(x), \qquad -1 \le x \le 1,$$

with $h_\pm(x) \in C^2[-1,1]$ and $h_+(\pm1) = h_-(\pm1) = h(\pm1)$.

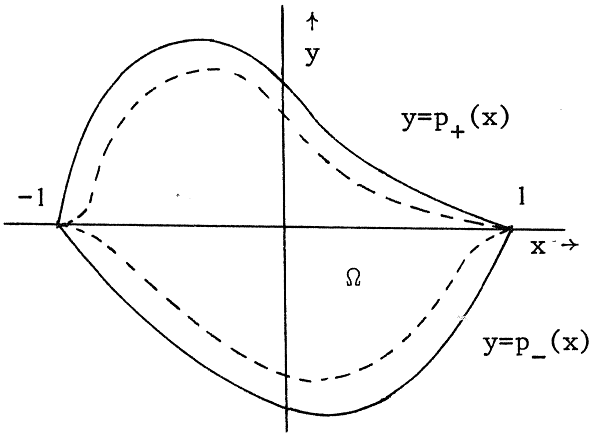


Fig. 1

<u>THEOREM 3.1</u>. *Let the function* $\phi(x,y;\varepsilon)$ *satisfy the differential equation* (3.4) *in the domain* $\Omega$*, as specified by* (3.5)*, with boundary conditions* (3.6). *For* $\varepsilon$ *sufficiently small there exists a positive constant* K *independent of* $\varepsilon$ *such that*

$$(3.7) \qquad |\phi - \phi_{as}| \le K\varepsilon \quad \textit{in} \quad \bar{\Omega}$$

*with*

(3.8) $$\phi_{as}(x,y;\varepsilon) = U(x) + V_0^+(x,y;\varepsilon) + V_0^-(x,y;\varepsilon),$$

(3.9) $$U(x) = \tfrac{1}{2}\{h(1) - h(-1)\}(x+1) + h(-1),$$

(3.10) $$V_0^\pm(x) = \bar{h}_\pm(x)\ \exp\{-\alpha_\pm(x)\frac{p_\pm(x)\mp y}{\varepsilon}\},$$

$$\bar{h}_\pm(x) = h_\pm(x) - U(x), \qquad \alpha_\pm(x) = \frac{p_\pm(x)}{p'_\pm(x)^2+1}.$$

PROOF. It is easily verified that in $y = p_\pm(x)\mp\varepsilon\eta$, $L_\varepsilon V_0^\pm$ does not decrease as $\varepsilon \to 0$ with $\eta$ fixed. Therefore, and because of the fact that we need a higher accuracy near $y = 0$, we introduce additional boundary layer terms:

(3.11) $$\Phi_{as}(x,y;\varepsilon) = U(x) + V_0^+(x,y;\varepsilon) + V_0^-(x,y;\varepsilon) +$$
$$+ \varepsilon\{V_1^+(x,y;\varepsilon) + V_1^-(x,y;\varepsilon)\} +$$
$$+ \varepsilon^2\{V_2^+(x,y;\varepsilon) + V_2^-(x,y;\varepsilon)\}$$

with $V_i^\pm = V_i^\pm(x,\eta)$, $\eta = \{p_\pm(x)\mp y\}/\varepsilon$ satisfying the following recurrent system of equations with boundary conditions

(3.12) $$M_0^\pm V_0^\pm = 0, \qquad V_0^\pm(x,0) = \bar{h}_\pm(x),$$

(3.13) $$M_0^\pm V_1^\pm = -M_1^\pm V_0^\pm, \qquad V_1^\pm(x,0) = 0,$$

(3.14) $$M_0^\pm V_2^\pm = -M_1^\pm V_1^\pm - M_2^\pm V_0^\pm, \quad V_2^\pm(x,0) = 0,$$

where

$$M_0^\pm \equiv \{p'_\pm(x)^2+1\}\frac{\partial^2}{\partial\eta^2} + p_\pm(x)\frac{\partial}{\partial\eta},$$

$$M_1^\pm \equiv 2p'(x)\frac{\partial^2}{\partial x\partial\eta} + \{p''_\pm(x)-\eta\}\frac{\partial}{\partial\eta},$$

$$M_2^{\pm} \equiv \frac{\partial^2}{\partial x^2}.$$

Note that $L_\varepsilon \equiv \varepsilon^{-1}M_0^{\pm} + M_1^{\pm} + \varepsilon M_2^{\pm}$. In (3.10) we already gave the solution of (3.12). For $V_1$ we find

$$(3.15)\qquad V_1^{\pm}(x,\eta) = [-\tfrac{1}{2}\bar{h}_{\pm}(x)\,\eta + \{\tfrac{1}{2}\bar{h}_{\pm}(x)p_{\pm}''(x) + \bar{h}_{\pm}'(x)p_{\pm}'(x)\}\eta^2]\exp\{-\alpha_{\pm}(x)\eta\},$$

while $V_2$ has the form

$$(3.16)\qquad V_2^{\pm}(x,\eta) = \sum_{i=1}^{4} A_i^{\pm}(x)\eta^i e^{-\alpha_{\pm}(x)\eta},$$

$$A_i^{\pm} = A_i^{\pm}(\bar{h}_{\pm},\bar{h}_{\pm}',\bar{h}_{\pm}'',p_{\pm},p_{\pm}',p_{\pm}'',p_{\pm}''').$$

From this we find by straight forward calculation that a constant M exists such that

$$(3.17)\qquad |L_\varepsilon\tilde{R}| \le M\varepsilon^2 \quad \text{in} \quad \Omega,$$

while also a constant N can be found such that $|\tilde{R}| \le N\varepsilon$ on $\partial\Omega$. From corollary 2.1 it is concluded that $|\tilde{R}| \le \tilde{K}\varepsilon$ in $\bar{\Omega}$ for some $\tilde{K}$. Finally, we prove the validity of (3.7) by verifying that the additional terms $V_i^{\pm}$, $i = 1,2$ are $O(\varepsilon)$ in $\bar{\Omega}$. □

It is noted that if $p_{\pm}(x) \in C^2$ and/or $h_{\pm}(x) \in C^1$ outside a neighbourhood of $x = \pm 1$, the estimate (3.17) can be replaced by $|L_\varepsilon\tilde{R}| \le (y^2+\varepsilon)M\varepsilon$ in $\Omega$. Then $V_2^{\pm}$ have to be multiplied with a smooth cut-off function in x which is identically one near $x = \pm 1$ and zero away from these points.

## 4. SOME REMARKS

We presented a method for estimating the accuracy of asymptotic approximations of problems of the type (1.1)-(1.9). The construction of such asymptotic solutions is an aspect of the problem we did not study in depth and which may lead to interesting procedures of matching local asymptotic solutions. Depending on the data and the required accuracy it may be necessary to construct corner layer solutions. Particularly for such approximations it is worthwhile to confine oneself to the lowest possible order of approximation needed to apply corollary 2.1. In this respect it is remarked that within a neighbourhood of the set $\Gamma$ the differential equation has to be satisfied with a higher order accuracy than elsewhere. This plays a role in problems with boundaries $\partial\Omega$ being smooth near $\Gamma$, in which a new type of corner layer of thickness $O(\sqrt{\varepsilon})$ occurs near $\partial\Omega \cap \bar{\Gamma}$.

The results of the above investigations are important in the study of dynamical systems with small random perturbations. Let us consider the n-dimensional system

$$(4.1) \qquad \frac{dz}{dt} = b(z)$$

with $b_i(z) = 0$ for $i = 1,\dots,k<n$ and with $(b_{j+k}(z),\dots,b_n(z))$ given by (1.7)-(1.8). Let the system be slightly disturbed by Gaussian white noise, then the trajectories of $z$ in $\Omega$ satisfy the stochastic differential equation

$$(4.2) \qquad dz_\varepsilon(t) = b(z_\varepsilon)dt + \varepsilon\sigma(z_\varepsilon)dw(t), \quad 0 < \varepsilon << 1,$$

where $w(t)$ is the n-dimensional Wiener process and $\sigma(z_\varepsilon)$ the diffusion matrix. A trajectory starting at a point $z_0 \in \Omega$ reaches the boundary $\partial\Omega$ with probability 1 (well-known result in probability theory). The probability distribution of points $z \in \partial\Omega$ where the trajectory first exits from the domain $\Omega$ is denoted by $p_\varepsilon(z,z_0)$. It can be shown (see MATKOWSKY and SCHUSS [7] that

$$(4.3) \qquad \phi(z;\varepsilon) = \int_{\partial\Omega} h(\bar{z})p_\varepsilon(\bar{z},z)dS_{\bar{z}} ,$$

where $\phi$ is the solution of the problem (1.1)-(1.7) with $z = (x,y)$ and

$$(4.4) \qquad \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} = \tfrac{1}{2}\sigma \times \sigma^*, \quad (^* \text{ denotes the adjoint}).$$

As an example we mention a problem from genetics, namely the problem of correlations between loci and the possible extinction of certain genotypes, see LITTLER [6]. A population of diploid individuals, each characterized by its genotype with respect to two loci and with two alleles at each locus, is described by the fractions of types AB, Ab, aB and ab. Let these fractions be $p_i$, $i = 1,2,3,4$ and let $\gamma$ be a measure for the total number of individuals ($\gamma >> 1$). Then in terms of the deterministic problem we have to deal with

$$(4.5)\qquad \frac{1}{2}\sum_{i=1}^{3} p_i(1-p_i)\frac{\partial^2\phi}{\partial p_i^2} - \sum_{i=1}^{3}\sum_{j=i+1}^{3} p_ip_j\frac{\partial^2\phi}{\partial p_i\partial p_j}$$

$$-\gamma\{p_1(1-p_1-p_2-p_3) - p_2p_3\}\left(\frac{\partial\phi}{\partial p_1} - \frac{\partial\phi}{\partial p_2} - \frac{\partial\phi}{\partial p_3}\right) = 0$$

in a domain $S = \{(p_1,p_2,p_3) \mid p_1,p_2,p_3 > 0,\ p_1+p_2+p_3 < 1\}$. Substitution of

$$(4.6)\qquad \begin{aligned} p_1 &= x_1x_2 + y \\ p_2 &= x_1(1-x_2) - y \\ p_3 &= (1-x_1)x_2 - y \end{aligned}$$

transforms equation (4.5) into

$$\frac{1}{2}\sum_{i=1}^{2} x_i(1-x_i)\frac{\partial^2\phi}{\partial x_i^2} + y\frac{\partial^2\phi}{\partial x_1\partial x_2} + \sum_{i=1}^{2} y(1-2x_i)\frac{\partial^2\phi}{\partial x_i\partial y} +$$

$$\{x_1x_2(1-x_1)(1-x_2) + y(1-2x_1)(1-2x_2) - y^2\}\frac{\partial^2\phi}{\partial y^2} - (\gamma+1)y\frac{\partial\phi}{\partial y} = 0.$$

The domain S transforms into a domain of the type $\Omega$ satisfying (1.3) with $\Gamma_x = \{(x_1,x_2) \mid 0 < x_1,x_2 < 1\}$. However, there is one complication: the operator $L_2$ is not uniformly elliptic. Thus, we are only able to make estimates in subdomains $\Omega'$ of $\Omega$ bounded away from $\partial\Omega$ with $\phi$ given on $\partial\Omega'$.

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